ELASTIC STABILITY OF PLATES UNDER NON-UNIFORM EDGE LOADINGS

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ELASTIC STABILITY OF PLATES UNDER

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This thesis is affectionately dedicated to my mother, my father, and my wife. Their love, patience, understanding and encouragement have made this work possible and meaningful.

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NOMENCLATURE

$2a$	Vertical height (depth) of plate
$2b$	Horizontal length (span) of plate
D	Flexural rigidity of plate = $Et^3/12(1-\mu^2)$
E	Modulus of elasticity
h_x, h_y	Finite-difference mesh spacing in the x and y directions, respectively
k	Buckling coefficient
N_x, N_y, N_{xy}	Normal and shearing forces in middle plane of plate, expressed in force per unit length
p	Maximum intensity of edge loading, expressed in force per unit area
t	Plate thickness
w	Deflection of middle plane of plate, perpendicular to the plate
x, y, z	Rectangular coordinates
∇^4	Biharmonic operator
μ	Poisson's ratio
ϕ	Airy's stress function
σ_x, σ_y	Normal components of stress parallel to x and y axes
τ_{xy}	Shearing stress in the xy plane

SUMMARY

The buckling loads of rectangular plates are an important factor in the design of many types of civil, mechanical, and aeronautical structures. One solution to a previously unsolved stability problem involving a square plate under one directional parabolic edge compression is presented in this thesis. The same loading condition is investigated for clamped and simply supported plates. Even though only one loading condition with a plate aspect ratio of one to one is considered, the method is quite general and can be used to solve partial and other nonhomogeneous edge loadings with no major changes in procedure.

The first step in the overall solution of this problem is the determination of the pre-buckling stress distribution in the plate. This part of the problem was solved numerically by approximating the compatability equation with a finite-difference equation and determining the value of the stress function at a number of points in the plate. The stress distribution is obtained using 400 points in the plate.

With the stress known at each point in the plate,

it is then possible to represent the plate deflection equation in finite difference form involving the stresses and the critical load intensity. The buckling load is obtained by generating the equations for varying load intensity and then finding the smallest value of load intensity for which the determinant of the system is zero.

Results indicate that a square plate parabolically loaded will sustain a lower total load than a uniformly loaded square plate. The entire problem is programmed in Algol for a digital computer with a core memory of at least 5000.

CHAPTER I

INTRODUCTION

The problem to be considered is the determination of the load at which buckling occurs in a square plate with simply supported or fixed edges and loaded as shown in Figure 1.

The theoretical elastic buckling load of a flat plate is the load required to maintain the originally straight plate in a slightly bent form. This load is important in design since buckling is the initial step in the changing plate configuration which leads to a stability type of failure.

Any mathematical solution to a stability problem requires that equilibrium, strain compatability, and boundary conditions be simultaneously satisfied. The equilibrium and compatability equations are combined into a single differential equation for the plate deflection which is given by Timoshenko (Reference 1, p.324) as:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} =$$

$$\frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right) \quad (1)$$

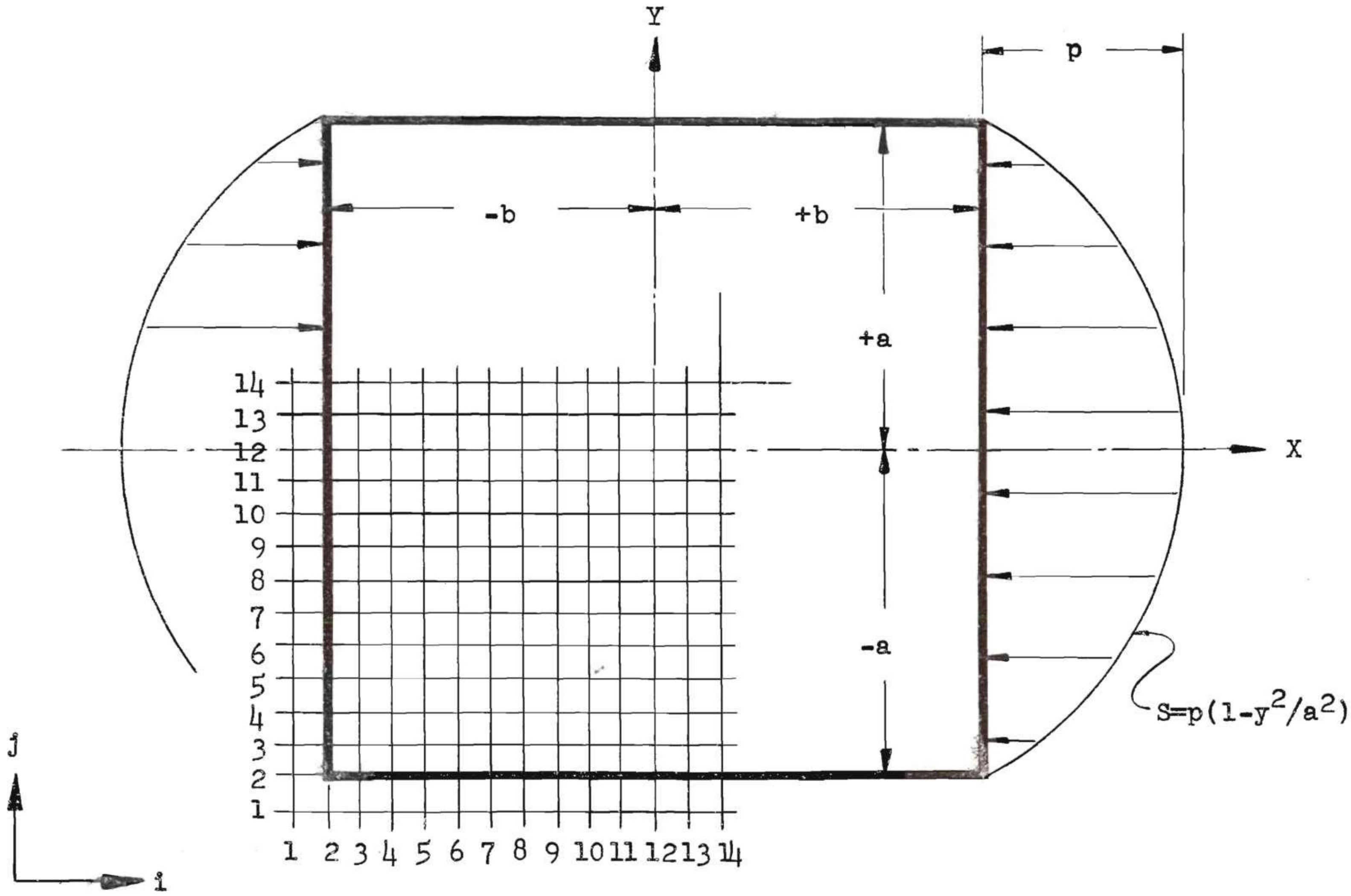


Figure 1. Square Plate Under Parabolic Edge Compression

where w = deflection perpendicular to the plane of the plate

$D = Et^3/12(1-\mu^2)$

μ = Poisson's ratio

N_x, N_y = Normal forces per unit of edge length acting in the x and y directions, respectively

N_{xy} = Shear forces per unit length in the xy plane

There are two basic methods of approach when working with equation 1. If possible, it is desirable to integrate the equation and use the particular boundary conditions at hand to determine the integration constants. This procedure yields exact answers, but unfortunately can be followed in only a few of the many problems which arise in practice.

An alternative approach involves the use of mathematical or numerical methods which do not completely satisfy the plate equation. These methods do not give exact answers, but the approximate answers are accurate enough for design work.

Equation 1 is most easily applied to cases with the forces N_x, N_y and N_{xy} constant throughout the plate. If the forces are not constant, the problem is more difficult since equation 1 becomes a variable coefficient differential equation.

For the plate considered here the forces are variable, and matters are further complicated because of the unknown force distribution within the plate. These two factors immediately make the exact integration of equation 1 impossible. One is therefore forced to the approximate schemes, an energy or finite difference approach.

It is felt that the main disadvantage of an energy approach is its lack of versatility. For example, if the supports and/or loads are discontinuous at the plate edges, an energy solution would be difficult, whereas a finite-difference approach is easily adaptable to any combination of support and loading conditions.

Using finite-difference equations, the problem must be treated as two separate problems -- determining the stress distribution in the plate, and then calculating its buckling load.

The first part is done by finding a stress function ϕ which satisfies the equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (2)$$

and the boundary conditions. After the stress function has been determined, the stresses may easily be calculated and used in the second part of the problem.

Equation 1 is then transformed into a system of linear deflection equations. This equation involving 13 adjacent points is written for each of the points within the plate. The load at which buckling occurs can be found by determining the load intensity which causes the determinant of the coefficients of the deflection equation to equal zero.

Both the pre-buckling stress distribution and the buckling load evaluations are approximate in that the finite-difference equations cannot possibly represent differential equations exactly. Accuracy depends quite heavily on the fineness of the subdividing mesh which in turn determines the number of points considered on the plate. This method of determining buckling loads has not been widely used due to the large amount of numerical work required, although Salvadori (Reference 2) and White (Reference 3) have done work in this field.

It is impossible to state exactly the magnitude of errors introduced by the two finite-difference operations involved in the complete solution. However, some general statements can be made.

The first operation which determines the stress distribution introduces a small error. For example, Reference 4 indicates good agreement between the finite-difference scheme and a more precise stress-function solution. The error between the finite-difference approach, which employed an 8 x 8 mesh, and the more precise method could have been reduced by using a finer mesh. The average error in stresses, with a 20 x 20 mesh, is probably reduced to less than three percent. (A comparison of Reference 10, p. 167, and Table A1 will help to substantiate this statement.)

The finite-difference buckling equation introduces an additional error. Part of this is caused from the obvious "finite-difference equation" approximation, and the remainder is the result of using "average" values of the stresses consistent with the smaller number of equations used in the determinant evaluation. The latter error is felt to be extremely small. White (Reference 3) has presented the errors in finite-difference solutions for buckling loads of plates under full edge compression. The errors as a function of mesh size are given in Table 1, in which n defines the size of an n x n subdividing mesh. Extrapolating this data for a 10 x 10 mesh, the size used in this presentation, indicates an error of approximately 3.3 percent. The total error in the buckling equation thus appears to be approximately four percent.

Assuming that all errors are of the same sign, the total maximum error will be approximately seven percent. With errors of opposite sign, it is evident that a more accurate solution would result.

Table 1. Errors in Finite-Difference Solutions

Plate	n	Percentage Error
Square	2	54.2
	3	31.3
	4	19.2
	5	12.9
	6	8.85

It is felt that the finite-difference approach holds the most advantages for this type of problem. While not as mathematically elegant as other methods mentioned, its simplicity and great flexibility in handling any type of boundary conditions are strong points in its favor. It is also readily adapted to digital computers.

There are several factors to be considered in applying finite-differences to this problem. The first point is whether to use forward, central, or backward differences. It is logical to use the central differences as they are invariably better for boundary value partial differential equation problems.

The second concerns the fineness of the subdividing mesh. By making the mesh finer and finer, more points on the plate are considered, resulting in more equations and increased accuracy.

The third has to do with the order of the finite-difference expressions. The basic difference between first and second order differences is that the remainder or error term which must be added to any difference expression is smaller for higher order type. Also as the order increases, the equation for each point has more terms involving adjacent points.

In general, increased accuracy can be obtained by either using a finer mesh or by using second order differences. Salvadori (Reference 6, p. 147) says that the choice of procedure depends upon the particular problem, but that only the use of a finer mesh can guarantee better accuracy.

Southwell (Reference 7) points out that the use of higher order differences tends to force the resulting curve or surface to have complete continuity in itself and its derivatives. This continuity is often not present in the physical problem; for example, the deflection curve and first two derivatives of a continuous three span beam are continuous, but the third and higher derivatives have discontinuities at the supports. He treats this point further and concludes that the higher order formulae "do not necessarily yield results of "increasing accuracy". On the other hand, Wang (Reference 8) mentions that higher order differences can certainly lead to increased accuracy.

The writer's decision to use a relatively fine mesh and first order differences may be questioned, but it is felt that this approach is most likely to lead to accurate results.

Using the method given here, a wide variety of buckling problems can be solved. This includes plates with variable moments of inertia, anisotropic plates, plates with any type of edge or body loadings, and irregularly shaped plates. However, this method is dependent entirely upon the use of a digital computer.

CHAPTER II

PLATE STRESS DISTRIBUTION

Stress Function Equations

The equation defining the stress function inside the plate is known as the compatability equation.

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \qquad (3)$$

After determining the function ϕ which satisfies equation 3 and the necessary boundary conditions, the stresses in the plate may be calculated from the expressions

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} \ ; \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2} \ ; \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} \ ; \qquad (4)$$

these relationships identically satisfy the two dimensional equilibrium conditions for the plate.

The finite-difference approximation to equation 3 is derived in the Appendix and appears as:

$$\nabla^4\phi \approx \frac{1}{h^4}\big(20\,\phi_{i,j} - 8(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}) + 2(\phi_{i+1,j+1} + \phi_{i-1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j-1}) + \phi_{i,j+2} + \phi_{i,j-2} + \phi_{i+2,j} + \phi_{i-2,j}\big)$$

where h = size of square grid, in inches

i, j = integer subscripts whose locations represent nodal points in the x and y directions respectively

In molecular form equation 3 becomes:

$$\nabla^4 \phi = \left(\frac{1}{h^4}\right) \begin{array}{ccccc} & & \boxed{1} & & \\ & \boxed{2} & \boxed{-8} & \boxed{2} & \\ \boxed{1} & \boxed{-8} & \boxed{20} & \boxed{-8} & \boxed{1} \\ & \boxed{2} & \boxed{-8} & \boxed{2} & \\ & & \boxed{1} & & \end{array} (\phi)$$

This equation or operator is applied to the stress function ϕ at each of the nodal points in the plate, thus producing a system of linear equations in ϕ in place of the biharmonic operator. The simultaneous solution of the system yields the values of ϕ at each nodal point, from which the stresses in the plate can be calculated. Also, derived in the Appendix are the approximations for equation 4 which are presented later in this chapter.

The Burroughs 220 has a limited storage capacity so the stress distribution was solved as a separate program. After the stress distribution was determined, the stresses were punched on data cards and used as input data for solving the buckling determinant.

Stress Function Boundary Conditions

Since the boundary loading conditions are specified, ϕ will be calculated to satisfy the applied loading. The basic method of analysis for treating boundary conditions is identical with that used by Timoshenko and Goodier (Reference 9, pp. 485-490). Figure 1 shows the problem to be considered (note that S is equal to the load intensity per unit area).

It is evident that no shear stress exists along any of the boundaries, σ_y along the edges $y = \pm a$ is zero, and at the edges $x = \pm b$, $\sigma_x = S$.

Using equation 4:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = p(1 - y^2/a^2)$$

Integration with respect to y yields:

$$\frac{\partial \phi}{\partial y} = p(y - y^3/3a^2) + f_1(x)$$

Integrating with respect to y again:

$$\phi = p(y^2/2 - y^4/12a^2) + y \cdot f_1(x) + f_2(x) \qquad (5)$$

Equation 5 is valid along the sides $x = \pm b$.

Also from equation 4:

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$$

Integration twice with respect to x:

$$\phi = x \cdot f_3(y) + f_4(y) \qquad (6)$$

Equation 6 is valid along the sides $y = \pm a$.

Equations 5 and 6 satisfy the condition $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \, \partial y} = 0$ on the edges if f_1 (x) and f_3 (y) are constant terms (not a function of x or y respectively).

Now the procedure for determining ϕ and its derivatives will be to start at any arbitrary point on the boundary and proceed around the plate, making use of the continuity which must exist at the corners. Knowing the equations for ϕ and its first derivatives, it will then be possible to determine the values of ϕ at all points on the boundary, and also at points just outside the boundary. These exterior points must be used in writing those finite-difference equations closest to the plate edges.

The stresses are independent of f_1 (double differentiation will elimate f_1); therefore it can be chosen arbitrarily at the starting point. Here f_1 is taken equal to zero at $x = -b$, $y = 0$, the starting point. Since a common point of the two boundaries exists at the corner, the values of ϕ are equated;

$$p(y^2/2 - y^4/12a^2) + \cancelto{0}{y\cdot f_1} + f_2(x) = x\cdot f_3 + f_4(y)$$

Therefore

$$f_4(y) = p(y^2/2 - y^4/12a^2) \; ; \qquad f_2(x) = x\cdot f_3$$

When evaluated at $x = -b$, $y = -a$

$$\phi = p(a^2/2 - a^2/12) + (-b)\cdot f_3$$

From equation 5

$$\phi = (-b) f_3 + p(a^2/2 - a^2/12)$$

and from equation 6 it is obvious that f_3 is an arbitrary constant, and may be set equal to zero. Equation 5 becomes

$$\phi = p(y^2/2 - y^4/12a^2) \qquad (7)$$

and equation 6 becomes

$$\phi = p(a^2/2 - a^4/12a^2) \qquad (8)$$

Figure 2 is obtained by using equations 7 and 8, and only one quarter of the plate need be considered since the plate is doubly symmetrical.

Thus the values of ϕ are known at all points on the boundary. To determine the first derivative of ϕ at the boundary we consider the vertical sides $x = \pm b$ where we know the first derivative $\frac{\partial \phi}{\partial x} = 0$.

If ϕ_{ext} represents an extended grid point and ϕ_{int} represents a nodal point immediatly inside the boundary,

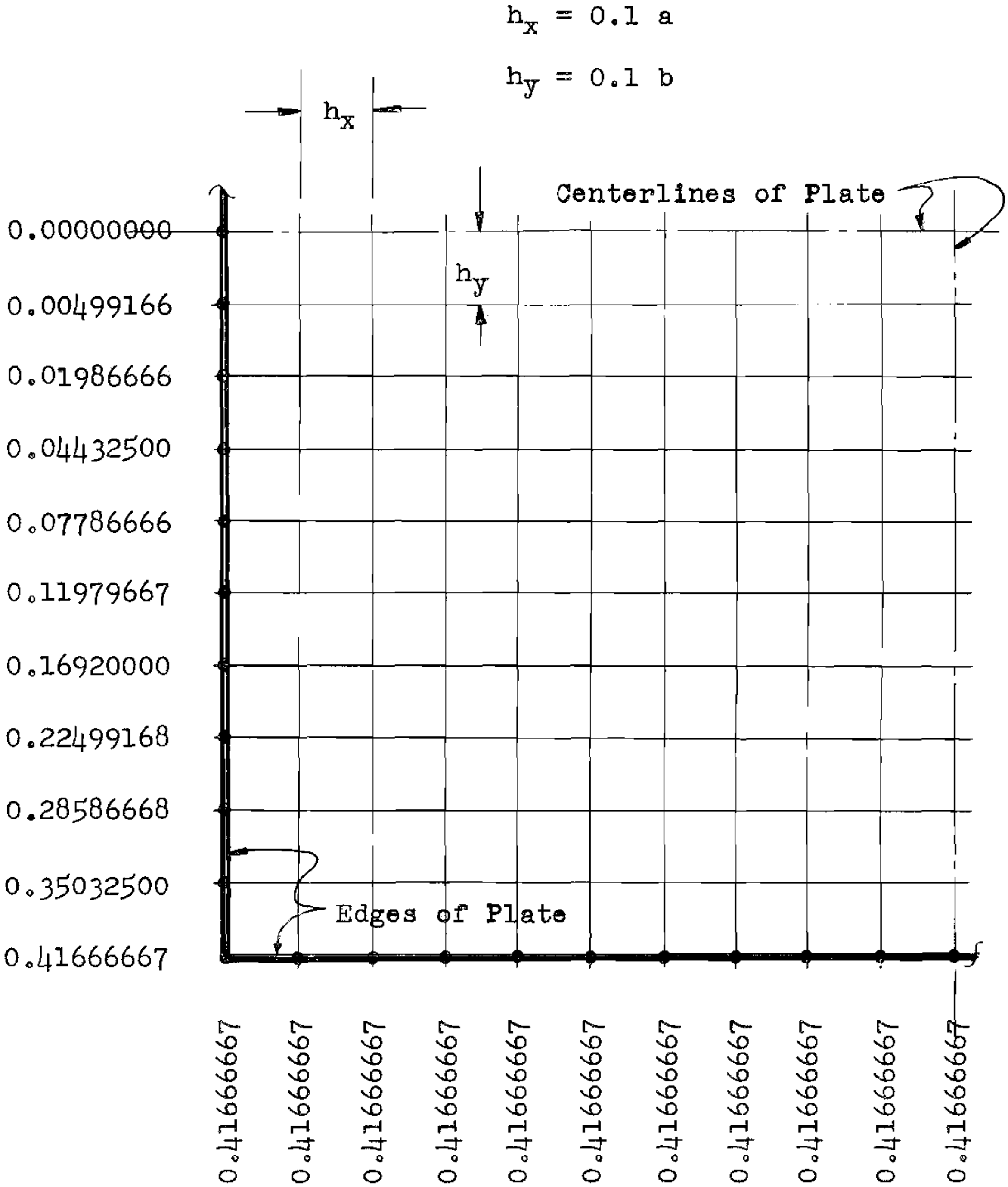


Figure 2. Stress Function Values (ϕ/pa^2) on the Boundary

it follows that

$$\left.\frac{\partial \phi}{\partial x}\right|_{\text{boundary}} = \frac{\phi_{ext} - \phi_{int}}{2\,h_x} = 0$$

therefore

$$\phi_{ext} = \phi_{int} \qquad (9)$$

on the sides $x = \pm\, b$.

Now differentiation of equation 7 yields

$$\frac{\partial \phi}{\partial y} = p(\,y - y^3/3a^2)$$

Evaluating this expression for $x = \pm\, b$ and $y = -\, a$ gives

$$\frac{\partial \phi}{\partial y} = p(-a + a/3\,) = -\,2\,pa/3$$

Realizing that the first derivative is equal to a constant value along the horizontal edges,

$$\frac{\phi_{ext} - \phi_{int}}{2\,h_x} = -\,2pa/3$$

Obviously

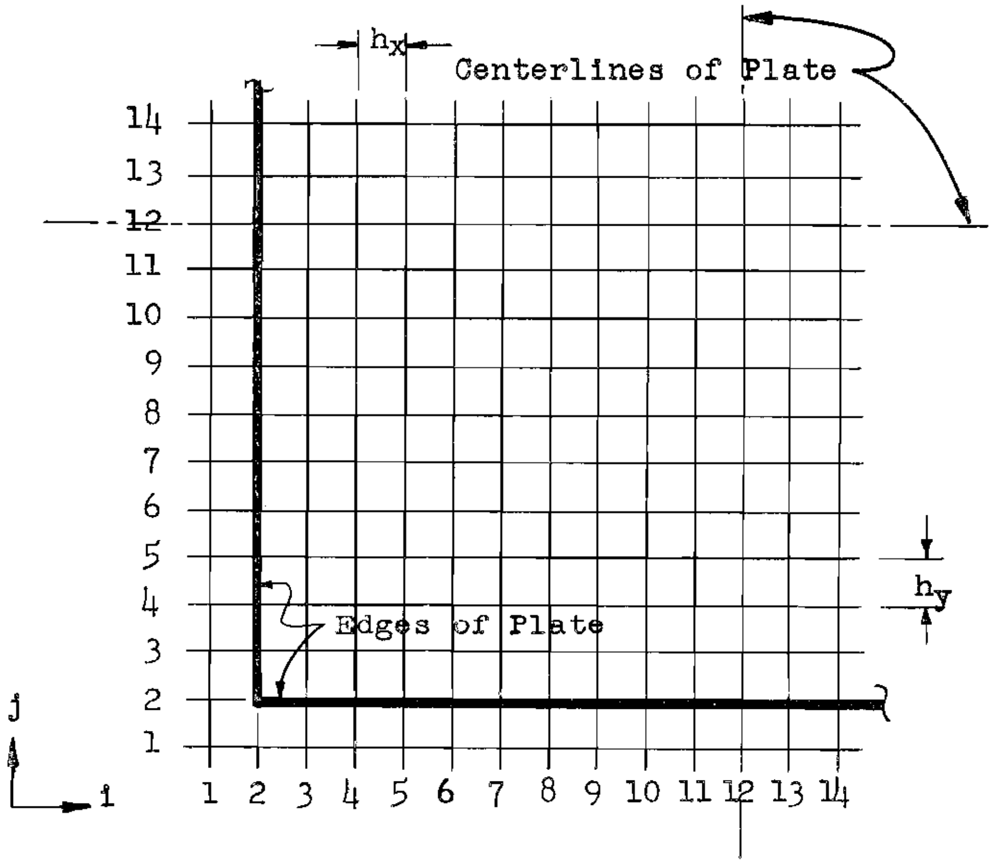
$$\phi_{ext} = \phi_{int} - 4pa\,h_x/3 \qquad (10)$$

For simplicity in calculations, a value of a = 10" is assigned. Equation 10 becomes $\phi_{ext} = \phi_{int} - 13.333$ along the sides $y = \pm a$, when p has been factored. For the other two sides (the centerlines of the plate) we may utilize the fact that this problem is doubly symmetrical. Thus $\phi_{ext} = \phi_{int}$ on the sides x = 0 and y = 0 so that the plate with boundary conditions assigned appears as shown in Figure 3.

Solution of Stress Function Equations

One quarter of the plate is divided into a mesh of 10 x 10 subdivisions. The set of 100 simultaneous equations is solved by a Gauss-Seidel relaxation procedure. After 1,000 sweeps over the 100 point mesh, the average residual is reduced to approximately $.001\phi$, requiring one hour of computer time. Obviously this method does not converge rapidly, but the simplicity of programming is not to be overlooked. Perhaps a computer faster than the Burroughs 220 would be more practical for this method if several different loading conditions had to be considered.

Other methods would give faster convergence, but are



For $3 \leq j \leq 12$:

$$\phi_{1,j} = \phi_{3,j} \; ; \; \phi_{14,j} = \phi_{10,j} \; ;$$

$$\phi_{13,j} = \phi_{11,j}$$

For $3 \leq i \leq 12$:

$$\phi_{i,1} = \phi_{i,3} - 13.333333 \; ;$$

$$\phi_{i,14} = \phi_{i,10} \; ; \; \phi_{i,13} = \phi_{i,11}$$

Figure 3. Stress Function Values of Extended Points

(See statements 8 through 31 in Table A1) more difficult to program. For example, the Alternating-Direction method as outlined in Reference 10 and Reference 11 would require approximately ten minutes for the same accuracy on the same computer.

The program used was more condensed than it appears in the Appendix because comments were added for the sake of clarity. Algol language is so versatile and logical that the general procedure can be understood even by those who have a minimum of programming experience.

Calculation of Stresses

The basic finite-difference equations for the stresses σ_x, σ_y and τ_{xy} are given by equation A-11. Since a square grid was chosen

$$\sigma_y = \frac{pa^2}{h_x^2}\left(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}\right) = \frac{pa^2}{h_x^2} \boxed{\begin{array}{|c|c|c|} 1 & -2 & 1 \end{array}} (\phi) \; ;$$

$$\sigma_x = \frac{pa^2}{h_y^2}\left(\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}\right) = \frac{pa^2}{h_y^2} \begin{array}{|c|} \hline 1 \\ \hline -2 \\ \hline 1 \\ \hline \end{array} (\phi) \; ;$$

$$\tau_{xy} = \frac{pa^2}{4h_xh_y}\left(\phi_{i+1,j-1} + \phi_{i-1,j+1} - \phi_{i+1,j+1} - \phi_{i-1,j-1}\right) =$$

$$\frac{pa^2}{4h_xh_y} \begin{array}{|c|c|c|} \hline 1 & & -1 \\ \hline & & \\ \hline -1 & & 1 \\ \hline \end{array} (\phi)$$

In the section on boundary values of ϕ, it was pointed out that the boundary values are given in terms of a constant multiplied by p a^2, and therefore all values of stresses must be multiplied by p a^2/h^2 to arrive at their true values.

The section 40 - 45 in Table A2 calculates the stresses from the previously determined values of ϕ.

CHAPTER III

DETERMINATION OF BUCKLING LOADS

The number of possible plate edge support conditions is very large if various combinations are considered. Plate edges can be simply supported, completely fixed, free, or partially restrained. In practice it is possible to have many combinations of the four basic conditions, and thus a complete coverage of all possibilities is beyond the scope of this thesis.

The edge conditions treated herein are restricted to two: all edges simply supported, and all edges fixed. The first type of support is chosen since it is a basic practical condition and also because it is a lower bound for all cases, except those with free edges. The second condition, completely fixed, was chosen as an upper bound for all possible cases. It is felt these two cases are of most general interest, and any cases with combinations of edge conditions or with partially restrained edges can be approximately determined by interpolation.

Buckling Equation

For flat plates subjected to in-plane edge loadings and no lateral or body forces, the equation of the buckled plate is:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{t}{D}\left[\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau_{xy}\frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right] \qquad (11)$$

If the stress distribution is constant throughout the plate, the right side can be simplified and the equation is much easier to apply. However, in this investigation the stress distributions are variable and equation 11 must be left in the form shown, with the stresses at each of the nodal points being an important factor in setting up the system of finite-difference approximations to equation 11.

Using the results of the Appendix, the second derivatives of w, expressed in finite-difference form, are:

$$\left.\begin{aligned}
\frac{\partial^2 w}{\partial y^2} &\approx \frac{1}{h_y^2}\left[w_{i,j+1} - 2w_{i,j} + w_{i,j-1}\right]; \\
\frac{\partial^2 w}{\partial x^2} &\approx \frac{1}{h_x^2}\left[w_{i+1,j} - 2w_{i,j} + w_{i-1,j}\right]; \\
\frac{\partial^2 w}{\partial x \partial y} &\approx \frac{1/4}{h_x h_y}\left[w_{i+1,j+1} + w_{i-1,j-1} - w_{i+1,j-1} - w_{i-1,j+1}\right]
\end{aligned}\right\} \qquad (12)$$

Letting $h_x = h_y = h$, equations 12 become

$$\left.\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &\approx \frac{1}{h^2}\begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline\end{array}\,(w)\ ; \\
\frac{\partial^2 w}{\partial y^2} &\approx \frac{1}{h^2}\begin{array}{|c|}\hline 1 \\ \hline -2 \\ \hline 1 \\ \hline\end{array}\,(w)\ ; \\
\frac{\partial^2 w}{\partial x\,\partial y} &\approx \frac{1}{4h^2}\begin{array}{|c|c|c|}\hline -1 & & 1 \\ \hline & & \\ \hline 1 & & -1 \\ \hline\end{array}\,(w)
\end{aligned}\right\} \qquad (13)$$

Using equation 13, the relationships given in the Appendix and in the section on stress function equations, equation 11 in finite-difference form is:

$$\left(\frac{1}{h^4}\right)\begin{array}{|c|c|c|c|c|}\hline & & 1 & & \\ \hline & 2 & -8 & 2 & \\ \hline 1 & -8 & 20 & -8 & 1 \\ \hline & 2 & -8 & 2 & \\ \hline & & 1 & & \\ \hline\end{array}\,(w) - \frac{t}{Dh^2}\left\{\sigma_x \begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline\end{array}\,(w) + \right.$$

$$\left.\sigma_y \begin{array}{|c|}\hline 1 \\ \hline -2 \\ \hline 1 \\ \hline\end{array}\,(w) + \frac{1}{2}\tau_{xy}\begin{array}{|c|c|c|}\hline 1 & & -1 \\ \hline & & \\ \hline -1 & & 1 \\ \hline\end{array}\,(w)\right\} = 0 \qquad (14)$$

where the stresses are the actual stresses due to a given loading. Since each of the stress values is multiplied by p (the load intensity per unit area) this factor p may be divided out, leaving the stresses as non-dimensional numbers. Also, since $p = N/t$, where N is the load per unit length along the plate edge, this substitution can be made. Then combining the constant factors, equation 14 becomes:

$$\begin{array}{|c|c|c|c|c|} \hline & & 1 & & \\ \hline & 2 & -8 & 2 & \\ \hline 1 & -8 & 20 & -8 & 1 \\ \hline & 2 & -8 & 2 & \\ \hline & & 1 & & \\ \hline \end{array} (w) - k \left\{ \text{SEE EQUATION 14} \right\} = 0 \quad (15)$$

where the constant k is defined as $(N/D) \cdot \frac{b^2}{(b/h)^2}$. The buckling load, $N_{x_{cr}}$, is thus given by

$$N_{x_{CR}} = k \frac{D\ (b/h)^2}{b^2} \quad (16)$$

For any given plate shape and loading case, the only unknown in equation 16 is the quantity k. Thus the determination of k enables the calculation of the buckling load from equation 16. It is also noted that equation 16 gives the buckling load in terms of a constant times the ratio D/b^2, which is the usual form for specifying critical loadings on flat plates.

The elements of the four finite-difference operators in equation 15 can be combined to give a single operator which will involve the deflection of the plate at point i, j, and at 12 adjacent points. The equation is:

$$\begin{array}{|c|c|c|c|c|}
 & & 1 & & \\
\hline
 & 2+K\left(\frac{T_{xy}}{2}\right) & -8-K(\sigma_y) & 2-K\left(\frac{T_{xy}}{2}\right) & \\
\hline
1 & -8-K(\sigma_x) & 20+2K(\sigma_x+\sigma_y) & -8-K(\sigma_x) & 1 \\
\hline
 & 2-K\left(\frac{T_{xy}}{2}\right) & -8-K(\sigma_y) & 2+K\left(\frac{T_{xy}}{2}\right) & \\
\hline
 & & 1 & &
\end{array} \; (w) = 0 \qquad (17)$$

Equation 17 applied to each nodal point inside the plate thus results in a set of simultaneous linear equations in terms of the deflection w. If $w = 0$, the equations are satisfied, but no buckling can exist with zero deflection. The only other possible manner in which the equations will be satisfied is if the determinant of the system is zero.

Therefore, to determine the buckling load, it is necessary to assume values of k, generate the buckling matrix by the use of equation 17, and evaluate the determinant. By choosing different values of k, the values of the resulting determinants will be close to zero (positive and negative values must be obtained) and the true value of k can be determined by interpolation.

Generation of Buckling Determinants

The buckling equations are in terms of constants which are the coefficients of equation 16. As mentioned previously, the system of equations generated must cover the entire plate. A 20 x 20 mesh would result in a 361 x 361 determinant. The evaluation of such a determinant would require a prohibitive amount of time on the Burroughs 220, and thus a coarser mesh is used to reduce the computing time. Also, as noted in Table 1, using n = 20 would improve the accuracy of the solution at most two percent. It is felt that the extra computations necessary for this larger determinant would introduce a greater round-off error in computer calculations, thus gaining approximately one percent more accuracy with 16 times the calculations.

The new mesh to be used is half as fine as the original mesh, or 10 x 10. A plate with this mesh is shown in

Figure 4. Since the deflections are all zero at the boundaries, it is necessary to consider only the 81 interior points. This results in an 81 x 81 determinant for each case considered.

It is now necessary to generate a 10 x 10 system of nodal point stresses from the previously obtained 20 x 20 system. A two dimensional interpolation scheme is used for this purpose. The new system is determined by considering the point i, j in Figure 5. This point is chosen to represent the area shown by the dotted lines around it. It was felt that this average value would be a more accurate approximation since it would tend to compensate for any large stress gradients which might occur near a nodal point.

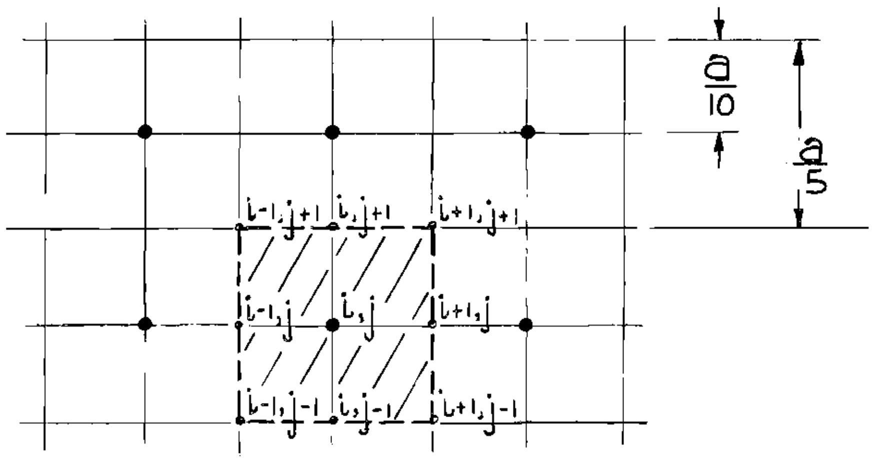


Figure 5. Interpolation Scheme

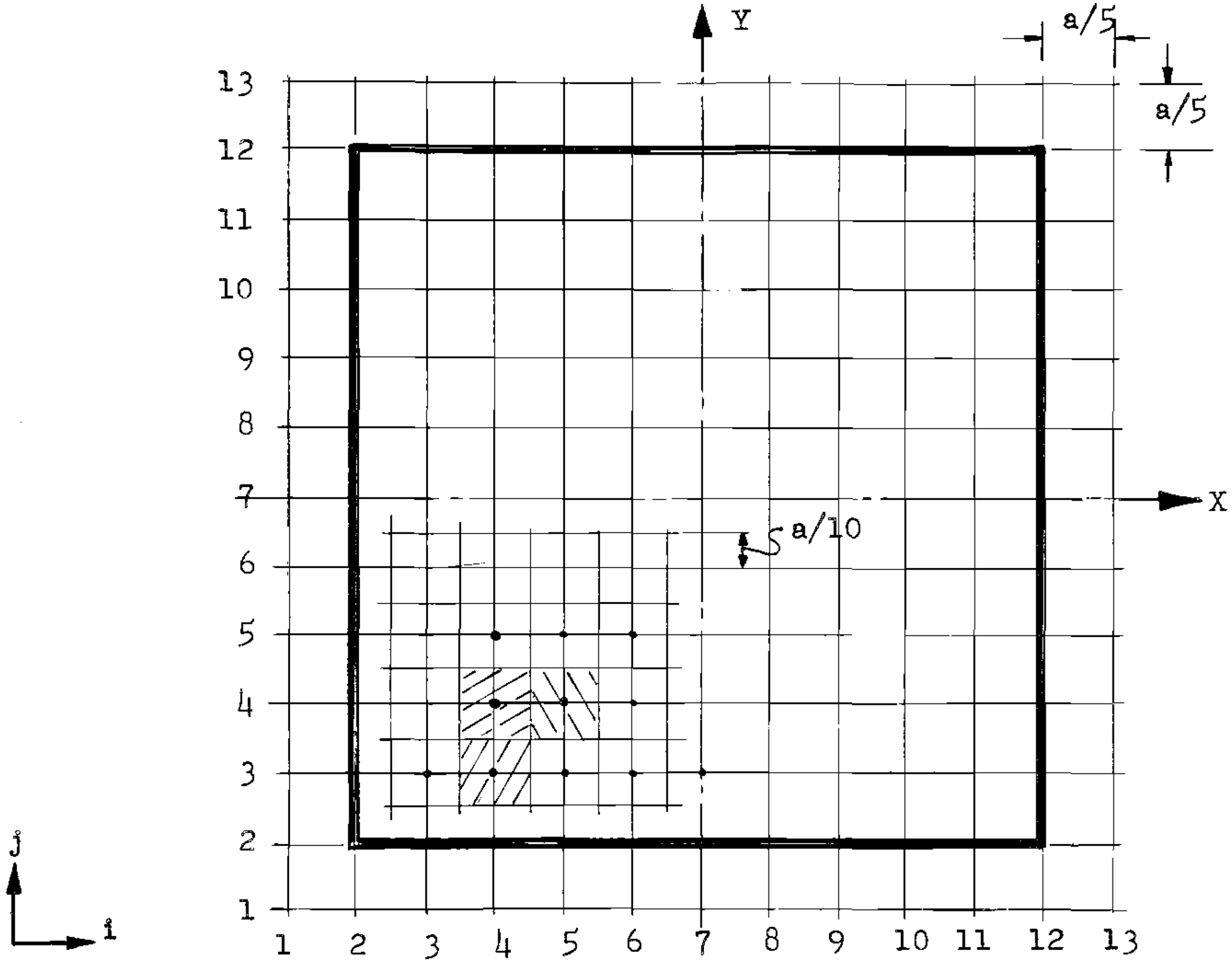


Figure 4. Plate Mesh for Buckling Calculations

Logically it follows that the value of the stresses at i, j is a function of the nine adjacent points.

$$\sigma_{i,j}\ \text{ave.} = \Big\{ 4\sigma_{i,j} + 2(\sigma_{i+1,j} + \sigma_{i-1,j} + \sigma_{i,j+1} + \sigma_{i,j-1}) + \sigma_{i+1,j+1} + \sigma_{i+1,j-1} + \sigma_{i-1,j+1} + \sigma_{i-1,j-1} \Big\} / 16 \qquad (18)$$

The above conversion is done by the computer. The program is short and can be found in the Appendix. The stresses in all four quadrants are obtained from symmetry considerations, keeping in mind that the shearing stresses are anti-symmetric with respect to the x and y axis.

A great deal of programming was necessary in order to evaluate an 81 x 81 determinant using a computer with only a 5000 core memory. The program used can be found in the Appendix. The determinant is reduced to an upper triangular form with the coefficients equal to one along the diagonal and zero below the main diagonal. The biharmonic operator when generated in matrix form will appear as in Figure 6 (a sparse matrix). Equation 17 is written first for the point i = 3, j = 3 and will appear in the first line of the determinant. The second line of the determinant is equation 17 written for the point $i = 3$, $j = 4$, etc. In order to accommodate the small storage space only the first 19 equations are stored in core memory. Then the coefficients in column 1 below the first number are reduced to zero with the first coefficient $A_{1,1} = 1.0$. Since the coefficients in row 1 do not influence the value of the determinant, they are now removed from memory and then the 20th equation is written in memory. This process is repeated until all elements are equal to one on the diagonal and zero below the main diagonal. Thus the value of the determinant will be equal to one. The factors required to obtain coefficients equal to one are stored in the computer and when they are multiplied together the value of the determinant is obtained.

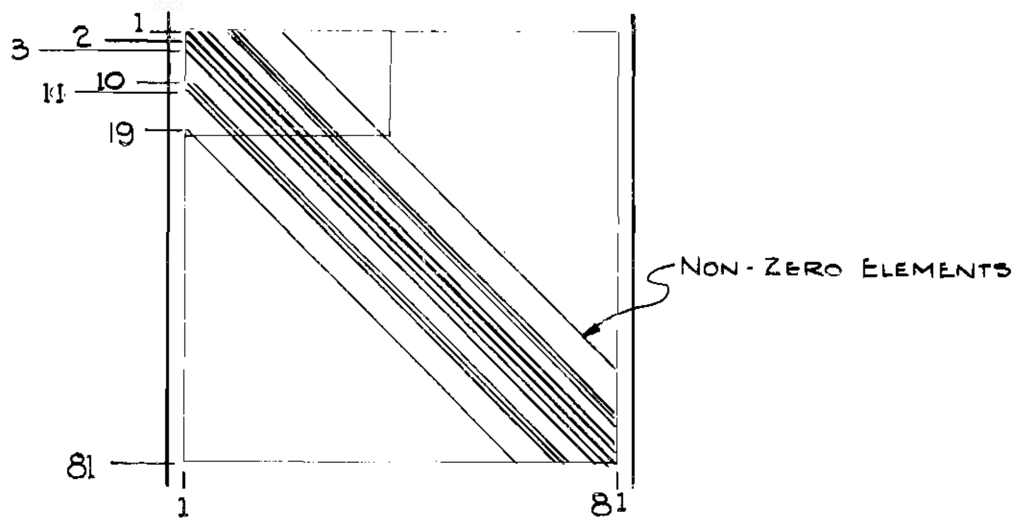


Figure 6. Coefficients in Buckling Determinant

Boundary Conditions

For simply supported plates, the moment at the edges is zero, i.e.,

$$M_x = \frac{1}{D}\left(\frac{\partial^2 w}{\partial x^2} + \mu \cancelto{0}{\frac{\partial^2 w}{\partial y^2}}\right) = 0$$

$$\therefore M_x \approx \frac{1}{h^2}\left(w_{i+1,j} - 2w_{i,j} + w_{i-1,j}\right) = 0$$

Since the deflection at the boundary is zero.

$$w_{i+1,j} - 2(0) + w_{i-1,j} = 0$$

$$\therefore \quad w_{i+1,j} = -w_{i-1,j}$$

Also, M_y yields the boundary condition

$$w_{i,j+1} = -w_{i,j-1}$$

Fixed edge plates have zero slope at their boundaries. We see at an x boundary

$$\frac{\partial W}{\partial x} = \frac{W_{i+1,j} - W_{i-1,j}}{2h} = 0$$

$$\therefore W_{i+1,j} = W_{i,j-1}$$

Similarly at a y boundary

$$\frac{\partial W}{\partial y} = 0$$

$$\therefore W_{i,j+1} = W_{i,j-1}$$

These conditions are incorporated in Table A3.

CHAPTER IV

RESULTS AND DISCUSSION OF RESULTS

The elastic buckling loads for the two cases considered are given in Table 2.

Table 2. Buckling Coefficients

PLATE	SUPPORTS	k	$N_{x_{cr}}$	k_{cr}	$N_{x_{cr}}$
Square	Hinged	0.44	$k \dfrac{D(b/h)^2}{(b)^2}$	4.51	$k_{cr} \dfrac{\pi^2 D}{(L)^2}$
Square	Fixed	1.02	$k \dfrac{D(b/h)^2}{(b)^2}$	10.4	$k_{cr} \dfrac{\pi^2 D}{(L)^2}$

As seen in Table 2 the buckling load found in this presentation is a function such that:

$$N_{x_{cr}} = \frac{k\ D\ (b/h)^2}{(b)^2}$$

where $N_{x_{cr}}$ is equal to the maximum applied load. A more familiar form for this quantity is:

$$N_{x_{cr}} = k_{cr} \frac{\pi^2 D}{(L)^2}$$

Since L = 2b we may convert k to k_{cr} i.e.,

$$k_{cr} = k \frac{(2b)^2}{\pi^2 h^2}$$

For the case considered h = 0.2b; therefore,

$$k_{cr} = k \frac{100}{\pi^2}$$

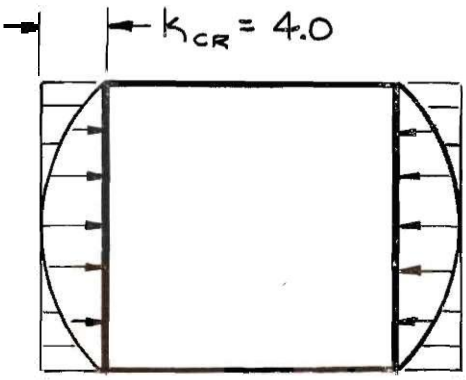
Since this investigation represents the first attempt to determine buckling loads for the loading considered, there are no available results to use for comparison and checking purposes. However, there are several factors which make the results seem reasonable and correct. They are:

a. A check case of uniform edge compression using the same procedure presented in this thesis gives answers quite close to the correct values.

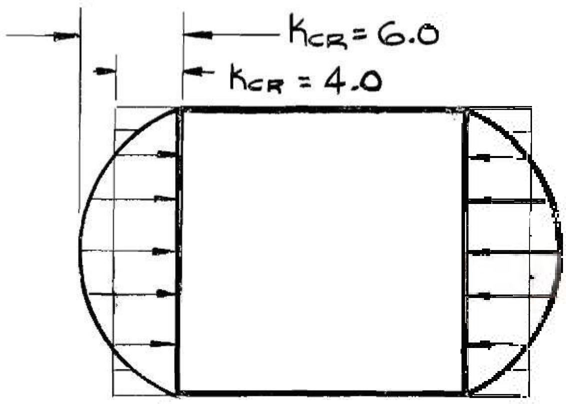
b. The critical loads exhibit a reasonable ratio between the fixed and simply supported edge conditions.

c. The plate stress distribution as solved in Chapter 2 compares favorably with that given by Timoshenko (Reference 10, p. 167). The determinant is evaluated by simple elimination which is an exact process; however, the answer is subject to round-off error. It is felt the final answers are correct, because both processes are accurate.

d. The buckling coefficients for square plates under uniform edge compression are $k_{cr} = 4.00$ and $k_{cr} = 8.33$ for simply supported and fixed edge conditions respectively. Suppose the values of k_{cr} are equal for uniform and parabolic loads, then we would suspect that k_{cr} is a function only of the maximum applied load (see Case 1, p. 37). If k_{cr} for the parabolically loaded plate is $1\frac{1}{2}$ times that of the uniformly loaded case, then this would indicate that k_{cr} is only a function of the total load (see Case 2, p. 37). Neither of these two suppositions is true because each is an oversimplification of the problem. Yet, we would expect, for the case of parabolic loading, to have $4 \leq k_{cr} \leq 6$ for the simply supported plate and $8.33 \leq k_{cr} \leq 12.5$ for the fixed edge plate. Both values determined here fall within these bounds.



CASE 1



CASE 2

APPENDIX

DERIVATION OF FINITE-DIFFERENCE EQUATIONS

For any continuous curve $y = f(x)$ or any continuous surface $w = f(x,y)$ it is possible to represent first and higher order derivatives by the method of finite differences. The basic principles of finite differences can be found in most books on numerical analysis; a particularly good treatment of the subject is found in Reference 7.

The plate problem reduces to a study of fourth-order partial differential equations. The representation of such equations in finite difference form will be derived here. The reference above should be consulted if a more extensive treatment of the subject is desired.

Consider the plate to be divided into a rectangular mesh of equal-sized rectangles, with the fineness of the mesh dependent upon the spacings h_x and h_y as shown in Figure A1. The intersections of the mesh lines shall be called nodal points and will be designated as shown in Figure A1.

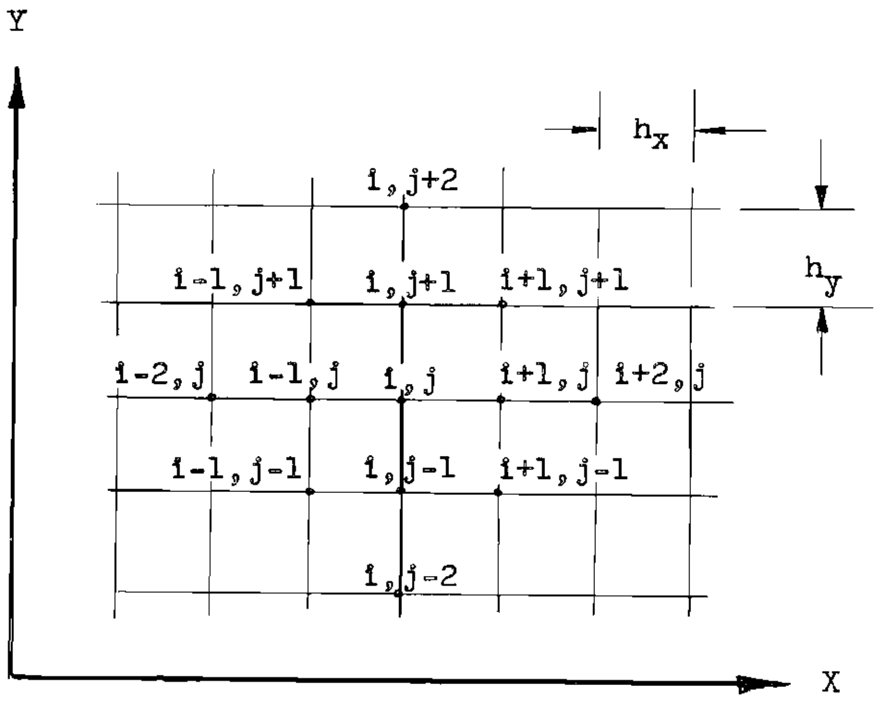


Figure A1 . Grid Orientation

The derivatives at (i,j) can be expressed in terms of the values of the function at (i,j), at the adjacent points, and in terms of the mesh dimensions. The x-derivatives can be represented as follows.

$$\left.\frac{\partial w}{\partial x}\right|_i \approx \frac{w_{i+1/2} - w_{i-1/2}}{h_x}$$

To avoid working with half mesh units, it is convenient to express this slope by:

$$\left.\frac{\partial w}{\partial x}\right|_i \approx \frac{w_{i+1} - w_{i-1}}{2h_x} \qquad \text{(A-1)}$$

The rate of change of slope, or $\frac{\partial^2 w}{\partial x^2}$, can be written as :

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{w_{i+1} - 2w_i + w_{i-1}}{h_x^2} \qquad \text{(A-2)}$$

Similarly $\frac{\partial^3 w}{\partial x^3}$ is given as:

$$\frac{\partial^3 w}{\partial x^3} \approx \frac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2h_x^3} \qquad \text{(A-3)}$$

Finally the fourth derivative becomes:

$$\frac{\partial^4 w}{\partial x^4} \approx \frac{1}{h_x^4}\left(w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}\right) \qquad \text{(A-4)}$$

By analogy the y-derivatives may be written

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_i \approx \frac{w_{j+1} - 2w_j + w_{j-1}}{h_y^2} \qquad \text{(A-5)}$$

and

$$\frac{\partial^4 w}{\partial y^4} \approx \frac{1}{h_y^4}\left(w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2}\right) \qquad \text{(A-6)}$$

The mixed derivatives are:

$$\frac{\partial^2 w}{\partial x \partial y} \approx \frac{1}{4\, h_x h_y}\left(w_{i+1,j+1} + w_{i-1,j-1} - w_{i+1,j-1} - w_{i-1,j+1}\right) \qquad \text{(A-7)}$$

and similarly

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} \approx \frac{1}{h_x^2\, h_y^2}\Big(w_{i+1,j+1} + w_{i-1,j+1} + w_{i+1,j-1} + w_{i-1,j-1}$$

$$-2w_{i+1,j} - 2w_{i,j+1} - 2w_{i,j-1} - 2_{i-1,j} + 4w_{i,j}\Big) \qquad \text{(A-8)}$$

The basic plate equation is the compatability equation

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \qquad \text{(A-9)}$$

Representing equation A-9 in finite-difference form by use of equations A-4, A-6, and A-8,

$$\nabla^4\phi \approx \left(\frac{6}{h_x^4} + \frac{6}{h_y^4} + \frac{8}{h_x^2 h_y^2}\right)\phi_{i,j} - 4\left(\left(\frac{1}{h_x^4}\right)(\phi_{i+1,j} + \phi_{i-1,j}) + \left(\frac{1}{h_x^2 h_y^2}\right)(\phi_{i,j+1} + \phi_{i,j-1} + \phi_{i+1,j} + \phi_{i-1,j}) + \left(\frac{1}{h_y^4}\right)(\phi_{i,j+1} + \phi_{i,j-1})\right) + 2\left(\left(\frac{1}{h_x^4}\right)(\phi_{i+2,j} + \phi_{i-2,j}) + \left(\frac{1}{h_x^2 h_y^2}\right)(\phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}) + \left(\frac{1}{h_y^4}\right)(\phi_{i,j-2} + \phi_{i,j+2})\right) = 0 \qquad \text{(A-10)}$$

In order to reduce equation A-10 a square grid may be used so that $h_x = h_y = h$, and after simplification:

$$\nabla^4\phi \approx \frac{1}{h^4}\left\{ 20\phi_{i,j} - 8(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}) + 2(\phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}) + \phi_{i+2,j} + \phi_{i-2,j} + \phi_{i,j+2} + \phi_{i,j-2} \right\} \qquad \text{(A-11)}$$

The equations for computing stresses are,

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\phi}{\partial y^2} \approx \frac{1}{h^2}(\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}) \\ \sigma_y &= \frac{\partial^2\phi}{\partial x^2} \approx \frac{1}{h^2}(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}) \\ \tau_{xy} &= -\frac{\partial^2\phi}{\partial x\,\partial y} \approx \frac{1}{h^2}(-\phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} - \phi_{i-1,j-1}) \end{aligned}\right\} \qquad \text{(A-12)}$$

with a square grid.

It should be pointed out that all finite-difference equations are approximations to the equations which they replace, and that higher order error terms must be included to be exact. Baron and Salvadori (Reference 6, pp. 168-172) show that this error is proportional to the square of the mesh dimensions, and thus can be nearly eliminated by using a sufficiently fine mesh.

TABLE A1. STRESS DISTRIBUTION

```
2
2COMMENT
2         ELASTIC STRESSES IN A FLAT PLATE SUBJECTED TO A PARABOLIC
2         COMPRESSIVE LOAD
2         ASPECT RATIO ONE TO ONE
2         PLATE SIZE 20 BY 20 (INCHES)
2         GRID SIZE 1 BY 1 (INCHES)
2                                                                          $ 1
2INTEGER I,J,N                                                             $ 2
2REAL OTHERWISE                                                            $ 3
2ARRAY    PHI(14,14),RES(14,14),SIGX(14,14),SIGY(14,14),TAUXY(14,14)       $ 4
2COMMENT
2         CONSIDERING ONE FOURTH OF THE PLATE WITH I AND J AS
2         LOCATED IN FIGURE 2
2
2         BEGIN ITERATIONS                                                 $ 5
2         FOR N=(1,1,1000)                                                 $ 6
2BEGIN                                                                     $ 7
2COMMENT
2         ASSIGNMENT OF BOUNDARY VALUES AND CONDITIONS                     $ 8
2         FOR I=(2,1,13)                                                   $ 9
2BEGIN    PHI(I,2)=-41.666667                          END                 $ 10
2         FOR J=(2,1,14)                                                   $ 11
2BEGIN    PHI(1,J)=PHI(3,J)                                                $ 12
2         PHI(13,J)=PHI(11,J)                                              $ 13
2         PHI(14,J)=PHI(10,J)                          END                 $ 14
2         FOR I=(1,1,14)                                                   $ 15
2BEGIN    PHI(I,14)=PHI(I,10)                                              $ 16
2         PHI(I,13)=PHI(I,11)                                              $ 17
2         PHI(I,1)=PHI(I,3)-13.33333333                END                 $ 18
2BEGIN    PHI(2,2)=-41.666667                                              $ 19
2         PHI(2,3)=-35.032500                                              $ 20
2         PHI(2,4)=-28.586668                                              $ 21
2         PHI(2,5)=-22.499168                                              $ 22
2         PHI(2,6)=-16.920000                                              $ 23
```

```
2          PHI(2,7)=-11.979167                                           $ 24
2          PHI(2,8)=-7.7866667                                           $ 25
2          PHI(2,9)=-4.4325000                                           $ 26
2          PHI(2,10)=-1.9866667                                          $ 27
2          PHI(2,11)=-0.49916667                                         $ 28
2          PHI(2,12)=0.00000000                                          $ 29
2          PHI(2,13)=-0.49916667                                         $ 30
2          PHI(2,14)=-1.9866667                 END                      $ 31
2COMMENT
2          CALCULATING A NEW VALUE OF PHI AT EACH OF THE 100 NODAL
2          POINTS FOR 1000 SWEEPS OVER THE GRID                          $ 32
2          FOR I=(3,1,12)                                                $ 33
2          FOR J=(3,1,12)                                                $ 34
2          PHI(I,J)=0.4(PHI(I+1,J)+PHI(I-1,J)+PHI(I,J+1)+PHI(I,J-1))
2          -0.1(PHI(I+1,J+1)+PHI(I-1,J+1)+PHI(I+1,J-1)+PHI(I-1,J-1))
2          -0.05(PHI(I+2,J)+PHI(I-2,J)+PHI(I,J-2)+PHI(I,J+2))      END  $ 35
2COMMENT
2          CALCULATE THE RESIDUAL AFTER 1000 SWEEPS                      $ 36
2          FOR I=(3,1,12)                                                $ 37
2          FOR J=(3,1,12)                                                $ 38
2          RES(I,J)=20.0(PHI(I,J))-8.0(PHI(I+1,J)+PHI(I-1,J)+PHI(I,J+1)
2          +PHI(I,J-1))+2.0(PHI(I+1,J+1)+PHI(I+1,J-1)+PHI(I-1,J+1)
2          +PHI(I-1,J-1))+PHI(I+2,J)+PHI(I-2,J)+PHI(I,J+2)+PHI(I,J-2)    $ 39
2COMMENT
2          CALCULATE THE STRESSES AFTER 1000 SWEEPS . NOTE THAT P/(H)(H)
2          HAS BEEN FACTORED  OUT OF EACH STRESS CALCULATION.            $ 40
2          FOR I=(2,1,12)                                                $ 41
2          FOR J=(2,1,12)                                                $ 42
2BEGIN     SIGX(I,J)=((PHI(I,J+1)-2.0(PHI(I,J))+(PHI(I,J-1))))           $ 43
2          SIGY(I,J)=-(PHI(I+1,J)-2.0(PHI(I,J))+PHI(I-1,J))              $ 44
2          TAUXY(I,J)=+((PHI(I+1,J+1)+PHI(I-1,J-1)-PHI(I-1,J+1)
2          -PHI(I+1,J-1))/(4.0))                   END                   $ 45
2          WRITE($$ANS1,FMT1)                                            $
2          WRITE($$ANS2,FMT1)                                            $
2          WRITE($$FMT2)                                                 $
2          WRITE($$ANS3,FMT1)                                            $
```

```
2          WRITE($$FMT2)                                                   $
2          WRITE($$ANS4,FMT1)                                              $
2          WRITE($$FMT2)                                                   $
2          WRITE($$ANS5,FMT1)                                              $
2OUTPUT    ANS1( FOR I=(2,1,12) $ FOR J=(2,1,12) $ SIGX(I,J))              $
2OUTPUT    ANS2( FOR I=(2,1,12) $ FOR J=(2,1,12) $ SIGY(I,J))              $
2OUTPUT    ANS3( FOR I=(2,1,12) $ FOR J=(2,1,12) $ TAUXY(I,J))             $
2OUTPUT    ANS4( FOR I=(3,1,12) $ FOR J=(3,1,12)  $ PHI(I,J))              $
2OUTPUT    ANS5( FOR I=(3,1,12) $ FOR J=(3,1,12)  $ RES(I,J))              $
2FORMAT    FMT1(*5*,B2,5F15.8,W0)                                          $
2FORMAT    FMT2(B80,W0)                                                    $
2          FINISH                                                          $
 I    J   PHI(I,J)       RES(I,J)        SIGX(I,J)        SIGY(I,J)       TAUXY(I,J)
 2    2  -41.666667      .00000000   -   .06499900        .00000000   -   .00000025
 2    3  -35.032500      .00000000   -   .18833500   -    .00376900   -   .00000350
 2    4  -28.586668      .00000000   -   .35833200        .00937800   -   .00000900
 2    5  -22.499168      .00000000   -   .50833200        .01402100   -   .00001275
 2    6  -16.920000      .00000000   -   .63833500        .00746500   -   .00001425
 2    7  -11.979167      .00000000   -   .14833200   -    .00712200   -   .00001392
 2    8  - 7.7866667     .00000000   -   .83833400   -    .02549060   -   .00001215
 2    9  - 4.4325000     .00000000   -   .90833340   -    .04373530   -   .00000965
 2   10  - 1.9866667     .00000000   -   .95833320   -    .05877590   -   .00000665
 2   11  -   .49916667   .00000000   -   .98833340   -    .06850838   -   .00000341
 2   12      .00000000   .00000000   -   .99833334   -    .07182811       .00004226
 3    2  -41.666667      .00000000   -   .06123000   -    .00000000   -   .00000425
 3    3  -35.030616      .00001000   -   .19679900        .00032700   -   .00068975
 3    4  -28.591364      .00027000   -   .35408500   -    .00663300   -   .00073725
 3    5  -22.506197      .00067500   -   .50273500   -    .01455000       .00483700
 3    6  -16.923765      .00107500   -   .63432000   -    .02411900       .01047975
 3    7  -11.975653      .00146000   -   .14644000   -    .03538300       .01403215
 3    8  - 7.7739817     .00184000   -   .83839300   -    .04734830       .01494052
 3    9  - 4.4107036     .00211100   -   .90993270   -    .05858030       .01339257
 3   10  - 1.9573582     .00231700   -   .96098420   -    .06767160       .00992443
 3   11  -   .46499701   .00259250   -   .99153650   -    .07353158       .00522312
 3   12      .03582775   .00237540   -  1.0014804    -    .07552561       .00014262
 4    2  -41.666667      .00000000   -   .05810000   -    .00000000   -   .00000875
```

4	3	-35.029059	.00046000	- .19797600	- .00281200	.00407250
4	4	-28.589427	.00122000	- .34888100	- .01241600	.00813625
4	5	-22.498676	.00217800	- .49548600	- .02342900	.01458625
4	6	-16.903411	.00329100	- .62861000	- .03392700	.02071525
4	7	-11.936756	.00438500	- .14384700	- .04353800	.02438432
4	8	- 7.7139484	.00533600	- .83918600	- .05210630	.02469005
4	9	- 4.3303269	.00611100	- .91367270	- .05933940	.02165855
4	10	- 1.8603781	.00663300	- .96686640	- .06484700	.01589527
4	11	- .35729577	.00747310	- .99860550	- .06827143	.00832512
4	12	.14718111	.00510720	- 1.0083832	- .06940093	.00023690
5	2	-41.666667	.00000000	- .04934300	- .00000000	- .00001175
5	3	-35.024690	.00085000	- .19236100	- .00374900	.01025125
5	4	-28.575074	.00230000	- .34226800	- .01229900	.01786250
5	5	-22.467726	.00422100	- .48875200	- .02203700	.02461800
5	6	-16.849130	.00624800	- .62378700	- .03091500	.02979825
5	7	-11.854321	.00819500	- .74229600	- .03826000	.03220887
5	8	- 7.6018087	.00990500	- .84131400	- .04403360	.03116635
5	9	- 4.1906108	.01128000	- .91913810	- .04836410	.02667725
5	10	- 1.6985510	.01222800	- .97483190	- .05136770	.01931555
5	11	- .18132310	.01381610	- 1.0079694	- .05311325	.01003678
5	12	.32793540	.00826790	- 1.0174003	- .05366045	.00029096
6	2	-41.666667	.00000000	- .03307900	- .00000000	- .00001550
6	3	-35.016572	.00142000	- .18194500	- .00287900	.01550150
6	4	-28.548422	.00374000	- .33446700	- .00870200	.02549750
6	5	-22.414739	.00656500	- .48287800	- .01513100	.03226975
6	6	-16.763934	.00955500	- .62049700	- .02069300	.03627750
6	7	-11.733626	.01243100	- .74231700	- .02482500	.03720637
6	8	- 7.4456355	.01496400	- .84488500	- .02756530	.03478677
6	9	- 4.0025306	.01700500	- .92593050	- .02920250	.02913852
6	10	- 1.4853562	.01839850	- .98405540	- .03007570	.02081768
6	11	.04776282	.02082180	- 1.0185317	- .03046181	.01072938
6	12	.56235014	.01152820	- 1.0274403	- .03055340	.00031117
7	2	-41.666667	.00000000	- .01105100	- .00000000	- .00001475
7	3	-35.005575	.00191000	- .16858500	- .00124100	.01856100
7	4	-28.513068	.00517000	- .32606000	- .00353600	.02968325
7	5	-22.346621	.00890600	- .47787100	- .00573200	.03615325

7	6	-16.658045	.01291700	- .61863700	- .00707900	.03910850
7	7	-11.588106	.01671100	- .74373000	- .00736200	.03884427
7	8	- 7.2618975	.02006900	- .84955800	- .00675950	.03545675
7	9	- 3.7852479	.02273000	- .93348740	- .00566420	.02920340
7	10	- 1.2420857	.02455350	- .99376600	- .00450720	.02062971
7	11	.30731055	.02781340	- 1.0293886	- .00365298	.01055670
7	12	.82731822	.01461350	- 1.0376540	- .00333894	.00030288
8	2	-41.666667	.00000000	.01345000	- .00000000	- .00001200
8	3	-34.993337	.00260000	- .15417100	.00053600	.01897050
8	4	-28.474178	.00651000	- .31775200	.0189800	.02991350
8	5	-22.272771	.01113900	- .47371300	.00410600	.03572375
8	6	-16.545077	.01598400	- .61784100	.00715900	.03782300
8	7	-11.435224	.02064200	- .74602900	.01087800	.03683475
8	8	- 7.0714009	.02473100	- .85472400	.01487750	.03308402
8	9	- 3.5623010	.02797300	- .94110690	.01867080	.02691750
8	10	- .99430800	.03019750	- 1.0031738	.02176890	.01884960
8	11	.57051126	.03422820	- 1.0397053	.02378005	.00958990
8	12	1.0956252	.01729490	- 1.0472821	.02446788	.00027090
9	2	-41.666667	.00000000	.03687700	- .00000000	- .00001050
9	3	-34.981635	.00306000	- .14058300	.00212000	.01680650
9	4	-28.437186	.00758000	- .31029000	.00675800	.02631400
9	5	-22.203027	.01297200	- .47040000	.01294800	.03109175
9	6	-16.439268	.01861000	- .61771100	.02002500	.03251600
9	7	-11.293220	.02393400	- .74860900	.02740400	.03128527
9	8	- 6.8957810	.02865300	- .85968200	.03450390	.02780460
9	9	- 3.3580249	.03237800	- .94803040	.04072980	.02243134
9	10	- .76829920	.03491910	- 1.0114946	.04555720	.01560787
9	11	.80993192	.03960440	- 1.0486987	.04858744	.00790392
9	12	1.3394643	.01938910	- 1.0556198	.04960210	.00021985
10	2	-41.666667	.00000000	.05606000	- .00000000	- .00000800
10	3	-34.972053	.00347000	- .12951300	.00333200	.01248575
10	4	-28.406952	.00844444	- .30438000	.01052500	.01948075
10	5	-22.146231	.01437600	- .46797400	.0198370	.02288425
10	6	-16.353484	.02053400	- .61788300	.03008800	.02376650
10	7	-11.178620	.02642500	- .75090900	.04037900	.02270347
10	8	- 6.7546659	.03154400	- .86376700	.04993700	.02004472

10	9	- 3.1944786	.035637	- .95355630	.05809180	.01608219
10	10	- .58784763	.03841450	- 1.0180183	.06428050	.01114010
10	11	- .58784763	.03841450	- .0180183	.06428050	.01114010
10	12	1.5337013	.02076840	- 1.0620471	.06937160	.00015450
11	2	-41.666667	.00000000	.06857300	- .00000000	- .00000425
11	3	-43.965803	.00384000	- .12230400	.00408500	.00663425
11	4	-28.387243	.00959000	- .30058900	.01288100	.00103365
11	5	-22.109272	.01633800	- .46648900	.02415770	.01210875
11	6	-16.297788	.02343900	- .61809500	.03642000	.01252925
11	7	-11.109399	.03027600	- .75247700	.04856400	.01192290
11	8	- 6.6634871	.03625500	- .86644900	.05969290	.01048967
11	9	- 3.0890241	.04101600	- .95711540	.06907173	.00838928
11	10	- .47167653	.04429920	- 1.0221780	.07612504	.00579480
11	11	1.1234931	.04999820	- 1.0600960	.08043950	.00291372
11	12	1.6585667	.02548460	- 1.0660842	.08185420	.00007920
12	2	-41.666667	.00000000	.07290700	.0000000	.00004450
12	3	-34.963638	.00363700	- .11980600	.00415200	.00014700
12	4	-28.380415	.00754000	- .29927800	.01306800	.00242750
12	5	-22.096470	.01200160	- .46598700	.02445500	.00029675
12	6	-16.278512	.01656300	- .61818800	.03277700	.00031300
12	7	-11.078742	.02075700	- .75230920	.04891300	.00030100
12	8	- 6.6320014	.02438200	- .86768100	.05999310	.00026652
12	9	- 3.0526413	.0271520-	.95834470	.6929840	.00021403
12	10	- .43162597	.02892880	- 1.0236078	.07626586	.00014895
12	11	1.1657816	.03377280	- 1.0616112	.08051400	.00007602
12	12	1.7015779	.00000190	- 1.0674488	.08188300	.00000012

TABLE A2. AVERAGE STRESSES

```
2INTEGER I,J,A,B                                                                 $
2REAL OTHERWISE                                                                  $
2ARRAY    SIGX(14,14),SIGY(14,14),TAUXY(14,14)                                   $
2         READ($$DATA1)                                                          $
2         READ($$DATA2)                                                          $
2         READ($$DATA3)                                                          $
2         FOR I=(1,1,14)                                                         $
2BEGIN
2         SIGX(I,13)=SIGX(I,11)                                                  $
2         SIGY(I,13)=SIGY(I,11)                                                  $
2         TAUXY(I,13)=TAUXY(I,11)                                                $
2                                                            END                 $
2         FOR J=(1,1,14)                                                         $
2BEGIN
2         SIGX(13,J)=SIGX(11,J)                                                  $
2         SIGY(13,J)=SIGY(11,J)                                                  $
2         TAUXY(13,J)=TAUXY(11,J)                                                $
2                                                            END                 $
2COMMENT
2         FINDING THE AVERAGE STRESSES AND CHANGING THEIR SUBSCRIPTS
2         TO REPRESENT THE ENTIRE PLATE INSTEAD OF ONLY ONE QUARTER OF
2         THE PLATE
2                                                                                $
2         A=0.00000000                                                           $
2         FOR I=(3,1,7)                                                          $
2BEGIN
2         B=0.00000000                                                           $
2         A=A+1                                                                  $
2         FOR J=(3,1,7)                                                          $
2BEGIN
2         B=B+1                                                                  $
2         SIGX(I,J)=(4.(+SIGX(I+A,J+B))
2         +(2)(SIGX(I+A+1,J+B)+ SIGX(I+A-1,J+B) + SIGX(I+A,J+B+1)
2         + SIGX(I+A,J+B-1))+ SIGX(I+A+1,J+B+1) + SIGX(I+A-1,J+B-1)
```

```
2          + SIGX(I+A+1,J+B-1) + SIGX(I+A-1,J+B+1))/16                 $
2          SIGY(I,J)=(4.(+SIGY(I+A,J+B))
2          +(2)(SIGY(I+A+1,J+B)+ SIGY(I+A-1,J+B) + SIGY(I+A,J+B+1)
2          + SIGY(I+A,J+B-1))+ SIGY(I+A+1,J+B+1) + SIGY(I+A-1,J+B-1)
2          + SIGY(I+A+1,J+B-1) + SIGY(I+A-1,J+B+1))/16                 $
2         TAUXY(I,J)=(4.(TAUXY(I+A,J+B))
2         +(2)(TAUXY(I+A+1,J+B)+TAUXY(I+A-1,J+B) +TAUXY(I+A,J+B+1)
2          +TAUXY(I+A,J+B-1))+TAUXY(I+A+1,J+B+1) +TAUXY(I+A-1,J+B-1)
2          +TAUXY(I+A+1,J+B-1) +TAUXY(I+A-1,J+B+1))/16                 $
2                                                   END                $
2                                                   END                $
2          FOR I=(3,1,7)                                               $
2          FOR J=(3,1,7)                                               $
2BEGIN
2           SIGX(I,14-J)=SIGX(I,J)                                     $
2           SIGX(14-I,J)=SIGX(I,J)                                     $
2           SIGX(14-I,14-J)=SIGX(I,J)                                  $
2           SIGY(I,14-J)=SIGY(I,J)                                     $
2           SIGY(14-I,14-J)=SIGY(I,J)                                  $
2           SIGY(14-I,J)=SIGY(I,J)                                     $
2          TAUXY(14-I,14-J)=+TAUXY(I,J)                                $
2          TAUXY(I,14-J)=-TAUXY(I,J)                                   $
2          TAUXY(14-I,J)=-TAUXY(I,J)                                   $
2                                                          END         $
2INPUT     DATA1( FOR I=(2,1,12) $ FOR J=(2,1,12) $ SIGX(I,J))        $
2INPUT     DATA2( FOR I=(2,1,12) $ FOR J=(2,1,12) $ SIGY(I,J))        $
2INPUT     DATA3( FOR I=(2,1,12) $ FOR J=(2,1,12) $TAUXY(I,J))        $
2          WRITE($$ANS1,FMT1)                                          $
2          WRITE($$FMT2)                                               $
2          WRITE($$ANS2,FMT1)                                          $
2          WRITE($$FMT2)                                               $
2          WRITE($$ANS3,FMT1)                                          $
2          WRITE($$FMT2)                                               $
2OUTPUT    ANS1( FOR I=(3,1,11) $ FOR J=(3,1,11) $ SIGX(I,J))         $
2OUTPUT    ANS2( FOR I=(3,1,11) $ FOR J=(3,1,11) $ SIGY(I,J))         $
2OUTPUT    ANS3( FOR I=(3,1,11) $ FOR J=(3,1,11) $TAUXY(I,J))         $
```

```
2FORMAT   FMT1(*5*,B2,5F15.8,W0)                                   $
2FORMAT   FMT2(B80,W0)                                              $
2         FINISH                                                    $
```

TABLE A3. BUCKLING DETERMINANT EVALUATION

```
2I,J_____INTEGER SUBSCRIPTS WHICH CORRESPOND TO FIGURE 4
2K,L_____ROW AND COLUMN OF DETERMINANT  RESPECTIVELY
2COMMENT THE ONLY CHANGE TO MAKE FOR THE SIMPLY SUPPORTED CASE IS TO
2        SET O=Q=-1.0                                                 $
2COMMENT
2        EVALUATION OF AN 81 X 81 BUCKLING DETERMINANT
2              SIMPLE ELIMINATION IS USED AND A STORAGE OF
2                  703 IS USED IN CORE MEMORY-EDGES FIXED             $
2INTEGER I,J,K,L,N,M                                                  $
2ARRAY   A(37,19),SIGX(13,12),SIGY(13,12),TAUXY(13,12)                $
2REAL OTHERWISE                                                       $
2        FOR P=(ANY VALUES)                                           $
2BEGIN                                                                $
2        READ($$DATA1)                                                $
2        READ($$DATA2)                                                $
2        READ($$DATA3)                                                $
2        M=0.000000000                                                $
2        C=1.0                                                        $
2COMMENT
2        ESTABLISHING THE COEFFICIENTS FOR THE ORIGINAL 37X19 MATRIX
2                   BIHARMONIC STEP                      (BLUE CARDS)
2                                 O R
2                                 I
2                                 I
2                     V O---------O-S-------O X
2                       I         I         I
2                       I         I         I
2               A O-----B-O---------O-C-------O-D-------O E
2                       I         I         I
2                       I         I         I
2                     Z O---------O-T-------O Y
2                                 I
2                                 I
2                                 O U
```

```
2          THE POINTS AT C
2                                                                              $
2          FOR K=(1,1,37)                                                      $
2          FOR L=(1,1,19)                                                      $
2BEGIN     A(K,L)=0.0        END                                               $
2          N=11                                                                $
2          FOR I=(3,1,5)                                                       $
2BEGIN     IF I EQL 5 $ N=3                                                    $
2          FOR J=(3,1,N)                                                       $
2BEGIN     O=0.0                                                               $
2          Q=0.0                                                               $
2          IF (J EQL 3) OR (J EQL 11) $ O=1.0                                  $
2          IF (I EQL 3) OR (I EQL 11) $ Q=1.0                                  $
2          K=(I-3)(9.0)+J-2                                                    $
2          L=(I-3)(9.0)+J-2                                                    $
2          A(K,L)=(20.0+Q+O+(P)(SIGX(I,J)+SIGY(I,J))(2))      END              $
2                                                             END              $
2COMMENT   THE POINTS AT A                                                     $
2          I=5                                                                 $
2          J=3                                                                 $
2          K=(I-5)(9.0)+J-2                                                    $
2          L=(I-3)(9.0)+J-2                                                    $
2          A(K,L)=1.0                                                          $
2COMMENT   THE POINTS AT E                                                     $
2          N=11                                                                $
2          FOR I=(3,1,5)                                                       $
2BEGIN     IF I EQL 5 $ N=3                                                    $
2          FOR J=(3,1,N)                                                       $
2BEGIN     K=(I-1)(9.0)+J-2                                                    $
2          L=(I-3)(9.0)+J-2                                                    $
2          A(K,L)=1.0                                   END                    $
2                                                       END                    $
2COMMENT   THE POINTS AT U                                                     $
2          FOR I=(3,1,4)                                                       $
2          FOR J=(5,1,11)                                                      $
2BEGIN     K=(I-3)(9.0)+J-4                                                    $
```

```
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=1.0                               END                  $
2COMMENT  THE POINTS AT S                                               $
2         N=10                                                          $
2         FOR I=(3,1,5)                                                 $
2BEGIN    IF I EQL 5 $ N=3                                              $
2         FOR J=(3,1,N)                                                 $
2BEGIN    K=(I-3)(9.0)+J-1                                              $
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=-(8.0+(P)(SIGY(I,J+1)))                     END        $
2                                                            END        $
2COMMENT  THE POINTS AT T                                               $
2         FOR I=(3,1,4)                                                 $
2         FOR J=(4,1,11)                                                $
2BEGIN    K=(I-3)(9.0)+J-3                                              $
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=-(8.0+(P)(SIGY(I,J-1)))                     END        $
2COMMENT  THE POINTS AT R                                               $
2         N=9                                                           $
2         FOR I=(3,1,5)                                                 $
2BEGIN    IF I EQL 5 $ N=3                                              $
2         FOR J=(3,1,N)                                                 $
2BEGIN    K=(I-3)(9.0)+J                                                $
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=1.0                               END                  $
2         END                                                           $
2COMMENT  THE POINTS AT V                                               $
2         N=10                                                          $
2         FOR I=(4,1,5)                                                 $
2BEGIN    IF I EQL 5 $ N=3                                              $
2         FOR J=(3,1,N)                                                 $
2BEGIN    K=(I-4)(9.0)+J-1                                              $
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=(2.0+(P)(TAUXY(I-1,J+1))(0.5))              END        $
2                                                            END        $
2COMMENT  THE POINTS AT X                                               $
```

```
2         N=10                                                          $
2         FOR I=(3,1,5)                                                 $
2BEGIN    IF I EQL 5 $ N=3                                              $
2         FOR J=(3,1,N)                                                 $
2BEGIN    K=(I-2)(9.0)+J-1                                              $
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=(2.0-(P)(TAUXY(I+1,J+1))(0.5))           END           $
2                                                         END           $
2COMMENT  THE POINTS AT Y                                               $
2         N=11                                                          $
2         FOR I=(3,1,5)                                                 $
2BEGIN    IF I EQL 5 $ N=3                                              $
2         FOR J=(4,1,N)                                                 $
2BEGIN    K=(I-2)(9.0)+J-3                                              $
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=(2.0+(P)(TAUXY(I+1,J-1))(0.5))           END           $
2                                                         END           $
2COMMENT  THE POINTS AT Z                                               $
2         FOR I=(4,1,4)                                                 $
2         FOR J=(4,1,11)                                                $
2BEGIN    K=(I-4)(9.0)+J-3                                              $
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=(2.0-(P)(TAUXY(I-1,J-1))(0.5))            END          $
2COMMENT  THE POINTS AT B                                               $
2         N=11                                                          $
2         FOR I=(4,1,5)                                                 $
2BEGIN    IF I EQL 5 $ N=3                                              $
2         FOR J=(3,1,N)                                                 $
2BEGIN    K=(I-4)(9.0)+J-2                                              $
2         L=(I-3)(9.0)+J-2                                              $
2         A(K,L)=-(8.0+(P)(SIGX(I-1,J)))                  END           $
2                                                         END           $
2COMMENT  THE POINTS AT D                                               $
2         N=11                                                          $
2         FOR I=(3,1,5)                                                 $
2BEGIN    IF I EQL 5 $ N=3                                              $
```

```
2          FOR J=(3,1,N)                                                   $
2BEGIN     K=(I-2)(9.0)+J-2                                                $
2          L=(I-3)(9.0)+J-2                                                $
2          A(K,L)=-(8.0+(P)(SIGX(I+1,J)))                         END      $
2                                                                 END      $
2COMMENT
2          REDUCING THE DETERMINANT TO AN UPPER TRIANGULAR MATRIX
2          WITH THE VALUE OF THE COEFFICIENTS EQUAL TO ONE ALONG
2          THE DIAGONAL AND ZEROS BELOW THE DIAGONAL                       $
2START..   C=C(A(1,1))                                                     $
2          FOR K=(2,1,37)                                                  $
2BEGIN     A(K,1)=(A(K,1))/(A(1,1))                               END      $
2          K=1                                                             $
2          A(K,1)=(A(K,1))/(A(1,1))                                        $
2          FOR L=(2,1,19)                                                  $
2BEGIN     IF A(1,L) EQL 0.0 $ GO TO POS1                                  $
2          C=(C)(A(1,L))                                                   $
2          FOR K=(2,1,37)                                                  $
2BEGIN     A(K,L)=(A(K,L))/-(A(1,L))                       END             $
2          K=1                                                             $
2          A(K,L)=(A(K,L))/-(A(1,L))                                       $
2          FOR K=(1,1,37)                                                  $
2BEGIN     A(K,L)=(A(K,L)+(A(K,1)))                   END                  $
2POS1..                                               END                  $
2COMMENT   ADDING ONE ROW OF COEFFICIENTS AT A TIME AND USING AN
2          ELIMINATION PROCEEDURE                                          $
2          N=4                                                             $
2          FOR I=(5,1,13)                                                  $
2BEGIN     IF I GTR 5 $ N=3                                                $
2          FOR J=(N,1,11)                                                  $
2BEGIN     M=M+1                                                           $
2          IF ABS (C) GTR 1.0**30 $ WRITE($$ANS,FMT)                       $
2          IF ABS (C) GTR 1.0**30 $ C=1.00000000                           $
2COMMENT   TO SHIFT EACH COEFFICIENT UP AND OVER ONE STEP                  $
2          FOR L=(2,1,19)                                                  $
2          FOR K=(2,1,37)                                                  $
```

```
2BEGIN    A(K-1,L-1)=A(K,L)                    END                    $
2         FOR K=(1,1,37)                                              $
2BEGIN    A(K,19)=0.00000000                          END             $
2         FOR L=(1,I,19)                                              $
2BEGIN    A(37,L)=0.00000000                          END             $
2         IF I GTR 11 $ GO TO NEXT                                    $
2         K=(I-3)(9.0)+J-2-M                                          $
2         L=(I-3)(9.0)+J-2-M                                          $
2         O=0.0                                                       $
2         Q=0.0                                                       $
2         IF (J EQL 3) OR (J EQL 11)  $ O=1.0                         $
2         IF (I EQL 11) $ Q=1.0                                       $
2         A(K,L)=(20.0+Q+O+(P)(SIGX(I,J)+SIGY(I,J))(2))               $
2POSS..   IF J EQL 11 $ GO TO POST                                    $
2         K=(I-3)(9.0)+J-1-M                                          $
2         A(K,L)=-(8.0+(P)(SIGY(I,J+1)))                              $
2POST..   IF J EQL 3 $ GO TO POSD                                     $
2         K=(I-3)(9.0)+J-3-M                                          $
2         A(K,L)=-(8.0+(P)(SIGY(I,J-1)))                              $
2POSD..   IF I EQL 11 $ GO TO POSB                                    $
2         K=(I-2)(9.0)+J-2-M                                          $
2         A(K,L)=-(8.0+(P)(SIGX(I+1,J)))                              $
2POSB..   IF I EQL 3 $ GO TO POSX                                     $
2         K=(I-4)(9.0)+J-2-M                                          $
2         A(K,L)=-(8.0+(P)(SIGX(I-1,J)))                              $
2POSX..   IF (I EQL 11) OR (J EQL 11) $ GO TO POSY                    $
2         K=(I-2)(9.0)+J-1-M                                          $
2         A(K,L)=(2.0-(P)(TAUXY(I+1,J+1))(0.5))                       $
2POSY..   IF (J EQL 3) OR (I EQL 11) $ GO TO POSV                     $
2         K=(I-2)(9.0)+J-3-M                                          $
2         A(K,L)=(2.0+(P)(TAUXY(I+1,J-1))(0.5))                       $
2POSV..   IF (I EQL 3) OR (J EQL 11) $ GO TO POSA                     $
2         K=(I-4)(9.0)+J-1-M                                          $
2         A(K,L)=(2.0+(P)(TAUXY(I-1,J+1))(0.5))                       $
2POSA..   IF (I EQL 3) OR (I EQL 4) $ GO TO POSR                      $
2         K=(I-5)(9.0)+J-2-M                                          $
```

```
2POSR..   IF (J EQL 10) OR (J EQL 11) $ GO TO POSE                                      $
2         A(K,L)=1.0                                                                    $
2         K=(I-3)(9.0)+J-M                                                              $
2         A(K,L)=1.0                                                                    $
2POSE..   IF (I EQL 10) OR (I EQL 11) $ GO TO POSU                                      $
2         K=(I-1)(9.0)+J-2-M                                                            $
2         A(K,L)=1.0                                                                    $
2POSU..   IF (J EQL 3) OR (J EQL 4) $ GO TO POSZ                                        $
2         K=(I-3)(9.0)+J-4-M                                                            $
2         A(K,L)=1.0                                                                    $
2POSZ..   IF (I EQL 3) OR (J EQL 3) $ GO TO NEXT                                        $
2         K=(I-4)(9.0)+J-3-M                                                            $
2         A(K,L)=(2.0-(P)(TAUXY(I-1,J-1))(0.5))                                         $
2NEXT..                                                                                 $
2         C=C(A(1,1))                                                                   $
2         FOR K=(2,1,37)                                                                $
2BEGIN    A(K,1)=(A(K,1))/(A(1,1))                                    END               $
2         K=1                                                                           $
2         A(K,1)=(A(K,1))/(A(1,1))                                                      $
2         FOR L=(2,1,19)                                                                $
2BEGIN    IF A(1,L) EQL 0.0 $ GO TO POS11                                               $
2         C=(C)(A(1,L))                                                                 $
2         FOR K=(2,1,37)                                                                $
2BEGIN    A(K,L)=(A(K,L))/-(A(1,L))                             END                     $
2         K=1                                                                           $
2         A(K,L)=(A(K,L))/-(A(1,L))                                                     $
2         FOR K=(1,1,37)                                                                $
2BEGIN    A(K,L)=(A(K,L)+(A(K,1)))                         END                          $
2POS11..                                         END                                    $
2         END                                                                           $
2         END                                                                           $
2POSF..   WRITE($$TITLE)                                                                $
2OUTPUT   ANS1(FOR L=(1,1,19) $ FOR K=(1,1,19) $ A(K,L),
2         FOR L=(1,1,19) $ FOR K=(19,1,37) $ A(K,L))                                    $
2FORMAT   FMT1(19(B2,19X6.2,W4),W4,19(B2,19X6.2,W4))                                    $
2         WRITE($$ANS,FMT)                                                              $
```

```
2OUTPUT   ANS(C)                                                                   $
2FORMAT   TITLE(B56,*GODZILLA*,W4)                                                 $
2FORMAT   FMT(B56,F20.10,W4)                                                       $
2INPUT    DATA1(FOR I=(3,1,11) $ FOR J=(3,1,11) $ SIGX(I,J))                       $
2INPUT    DATA2(FOR I=(3,1,11) $ FOR J=(3,1,11) $ SIGY(I,J))                       $
2INPUT    DATA3(FOR I=(3,1,11) $ FOR J=(3,1,11) $ TAUXY(I,J))                      $
2LAST..                                           END                              $
2         FINISH                                                                   $
```

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